Journal of Computational Physics **148,** 605–620 (1999)
Article ID jcph.1998.6132, available online at http://www.idealibrary.com on IDEAL®

An Improved Finite-Element Flux-Corrected Transport Algorithm

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Received February 3, 1998; revised October 15, 1998

An improved finite-element flux-corrected transport (FE-FCT) method for the numerical solution of hydrodynamic conservation equations is described, based on the method developed by Lohner and his collaborators to solve conservation equations in fluid mechanics, and its application is extended to gas discharge problems. The high- and low-order schemes used are the ones proposed by Lohner who adds diffusion to the high-order scheme by subtracting the lumped-mass matrix from the consistent-mass matrix to give the low-order scheme; the diffusion coefficient is adjusted globally. A variable diffusion coefficient is introduced; it is assumed to be constant in each element and is shown to transform the high-order solution to a scheme equivalent to an upwind scheme which has minimal diffusion but ensures positive results. This avoids the complexity of upwinding in FE, especially in two dimensions. It is also shown that the correct amount of "real" diffusion may be easily added to the algorithm when required, for example, for electrons. Results are presented which show that the high-order scheme reduces to the upwind difference scheme when the new diffusion is used. The proposed FCT scheme is shown to give similar results, in comparison with a fourth-order FD-FCT algorithm. Finally, the new method is applied to a streamer propagation problem in one dimension, and the results obtained are shown to agree well with previously published results.

Key Words: flux-corrected transport; conservation equations; streamers; gas discharges; Poisson's equation.

1. INTRODUCTION

Equations describing the drift and diffusion of charged particles in an electric field represent the starting point for most theoretical studies in gaseous discharges. In many cases the electric field, being controlled by space-charge effects, must be obtained from Poisson's

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equation. In these cases the field varies strongly in both space and time, and numerical methods are needed to account for charge cancellation in an evaluation of the net charge density. Near the electrodes the electric field variation is particularly great, which demands a very fine spatial mesh, whereas the body of the discharge plasma rarely exhibits the steep gradients associated with the electrodes. A nonuniform spatial mesh is therefore essential for an accurate numerical treatment of electrode phenomena and the body of the gas simultaneously. In addition, the numerical algorithm should fulfil the following requirements:

- it should give positive, accurate results, free from nonphysical density fluctuation and numerical diffusion;
- it should be computationally efficient; and
- it should be easily extendable to two dimensions.

The first requirement is fullfilled by using a very accurate method such as the flux corrected transport (FCT) which introduces the “real” diffusion in the system. In the field of gas discharges finite difference methods (FD) are often used in preference to finite elements (FE). However, in order to fulfil the last two requirements one has to resort to FE as they offer computational efficiency through the use of unstructured grids, and they can be easily extended to two dimensions. All these suggest that a finite element version of FCT should be developed.

The FE-FCT method presented in this paper is an extension of the method proposed by Lohner *et al.* [7] which has been used very successfully in fluid mechanics in two dimensions, and it has the distinct advantage, in comparison with other FE schemes, that no operation splitting for multidimensional problems is required. Lohner uses the two-step Lax–Wendroff method as the high-order scheme and adds diffusion to transform it to a low-order one. Diffusion is added by subtracting the lumped mass matrix from the consistent mass matrix and the diffusion coefficient is taken as constant everywhere.

This kind of diffusion fails to satisfy the optimal condition which is critical to the performance and accuracy of the FCT method. Since the gas discharge calculations involve variable speed and mesh, if at a certain region the diffusion coefficient is low, then oscillations will result; if the diffusion coefficient is higher than the optimum then the results will be unrealistically diffusive. It is, however, not possible to have optimum diffusion coefficient everywhere at any time with a fixed global diffusion coefficient.

Lohner *et al.* [7], do not strictly follow Zalesak’s method in that they use a fixed diffusion coefficient in order to derive a low-order solution, rather than using a low-order algorithm that gives positive, ripple-free, results. Thus the approach taken by Lohner *et al.* [7] is very similar to that of Boris and Book [2], who made a detailed study of the effect of the different choices of diffusion coefficients to control phase and diffusion errors.

We choose to follow the approach of Zalesak [18] more strictly, where both high- and low-order solutions are computed, as well as the flux necessary to transform the high-order solution into the low-order solution. In this case there is no scope to adjust the diffusion coefficients.

In order to adapt Lohner’s method to the strict Zalesak formulation we need to develop a suitable low-order solution. The low-order solution most often used is the upwind difference method [14], which has the minimum diffusion that guarantees positive results [4].

Below we show, both mathematically and numerically, that we can add precisely the correct variable diffusion to transform our high-order solution into a low-order solution, which is equivalent to an upwind difference solution. Then we follow Zalesak’s method of developing an FCT algorithm. This method has the added advantage of giving a ripple-free positive low-order solution, without having to consider upwind fluxes which can be computationally expensive in two dimensions.

The variable diffusion coefficient used, is constant within each element, and is self-adjusted depending on the element size, particle speed, and time step. The optimal diffusion coefficient is found to be the diffusion coefficient inherent in the upwind method (Godunov [4], Boris and Book [1], Steinle and Morrow [14], Ward [16]); this is the minimum diffusion needed to guarantee positive results. Nevertheless, if the upwind algorithm was used as the low-order scheme the FCT would become very computationally expensive and complex, in two dimensions. So, the effective diffusion coefficient for the upwind method is used, and added to the high-order scheme as a mass diffusion in order to give the low-order scheme. The computational efficiency of Lohner's method is maintained, whereas the optimum variable diffusion we introduce in this paper ensures improved FCT performance.

Initially the finite difference upwind method (FD-UW) is compared with the new low-order scheme, and the results are shown to be almost identical. Several tests are conducted to show that the method works for various conditions. The improved FE-FCT is then compared with Lohner's method in one dimension, the Lax–Wendroff FD-FCT, and a fourth-order FD-FCT [14]; it is shown to exhibit improved performance over the first two, and very similar performance to the third.

Finally, the method is applied to problems which involve diffusion, and to a positive streamer calculation. The results are in very good agreement with results obtained using the fourth-order FD-FCT method by Morrow for the same problem [11].

2. FE-FCT

The details of the FE-FCT method proposed by Lohner can be found in the literature [6–9, 13]. The authors will concentrate on the diffusion term which is the parameter of importance. Consider the simple advection equation in 1D,

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \tag{1}$$

where u is the speed, f is the flux, t is the time, and x is the distance. The present FCT algorithm uses the two-step form of the one-step Taylor–Galerkin scheme described in [3] as the high-order scheme. This scheme belongs to the Lax–Wendroff family. Given the equation above the solution is advanced from t^n to $t^{n+1} = t^n + \Delta t$ in two steps:

First step (*advective predictor*),

$$u^{n+1/2} = u^n - \frac{\Delta t}{2}\frac{\partial f^n}{\partial x}. \tag{2}$$

Second step (*corrector step*),

$$\Delta u^n = u^{n+1} - u^n = -\Delta t \frac{\partial f^{n+1/2}}{\partial x}. \tag{3}$$

The spatial discretization of Eqs. (2) and (3) is then performed via the classic Galerkin weighted residual method using linear elements, and the following system of equations is obtained [7]:

$$\mathbf{M_c}\boldsymbol{\Delta}\mathbf{u^n} = \mathbf{R^n}, \tag{4}$$

where $\mathbf{M_c}$ denotes the consistent mass matrix [8], $\boldsymbol{\Delta}\mathbf{u^n}$ is the vector of nodal increments, and $\mathbf{R^n}$ is the vector of added element contributions to the nodes.

The next step for the FE-FCT method is to derive the low-order scheme. The requirement placed on the low-order scheme in any FCT method is monotonicity and strictly positive density. The low-order scheme must not produce any nonphysical or numerical wriggles. The better the low-order scheme, the easier the resulting task of limiting will be. Lohner added mass diffusion to the lumped-mass Taylor–Galerkin scheme in the context of FE-FCT. This simplest and least expensive form of diffusion is obtained by subtracting the lumped-mass matrix [8] from the consistent mass matrix for linear elements

$$\mathbf{Diff} = c_d(\mathbf{M_c} - \mathbf{M}_\mathrm{l})\mathbf{u^n}, \tag{5}$$

where c_d is the diffusion coefficient and is assumed to be constant within the range 0–1. This diffusion is added to the high-order scheme to give a low-order scheme. The diffusion cannot be simply added to the high-order scheme in order to obtain monotonic results, as a multipoint coupling of the right-hand side occurs due to the consistent mass matrix applied to the high-order scheme [7]. The imposition of monotonicity can nevertheless be achieved by using the lumped-mass matrix instead. As the terms originating from the discretization of the fluxes in Eq. (4) are the same as in the low-order scheme, the low-order scheme is given by

$$\mathbf{M}_\mathrm{l}\boldsymbol{\Delta}\mathbf{u}^\mathrm{l} = \mathbf{R} + c_d(\mathbf{M_c} - \mathbf{M}_\mathrm{l})\mathbf{u^n}. \tag{6}$$

The final step of the FE-FCT is the flux-limiting procedure provided by Zalesak's multidimensional limiter [18].

3. DIFFUSION COEFFICIENT IMPROVEMENT

As the optimal diffusion coefficient varies with speed, mesh size, and step size, it is clear that a global diffusion coefficient cannot be optimal everywhere, especially in the case of gas discharge calculation where one has variable speed and mesh size. In Fig. 1 we demonstrate the problem with Lohner's method and show the improvement that can be obtained by the new method using as an example the propagation of a square wave of amplitude 10 and width 0.2 cm, starting from $x = 0.3$ cm with a linearly decreasing speed of the form $w = (5 - x)$ cm/s. The figure shows the propagation of the wave after 1 s with three different diffusion coefficients, $c_d = 0.4$, $c_d = 1$ constant everywhere (Lohner's method), and a variable self-adjusted diffusion coefficient, c_d (the authors' method). In the case where $c_d = 0.4$, at some points the diffusion is too low and as a result the FCT gives oscillations. On the other hand, when $c_d = 1$ the opposite happens; the diffusion is more than enough and, hence, the FCT result is diffusive. Only when the self-adjustable diffusion coefficient is used are the results found to be optimal.

The upwind scheme is the ideal low-order scheme because it ensures that the results have minimum diffusion and are always positive (Godunov [4], Lohner *et al.* [7], Steinle

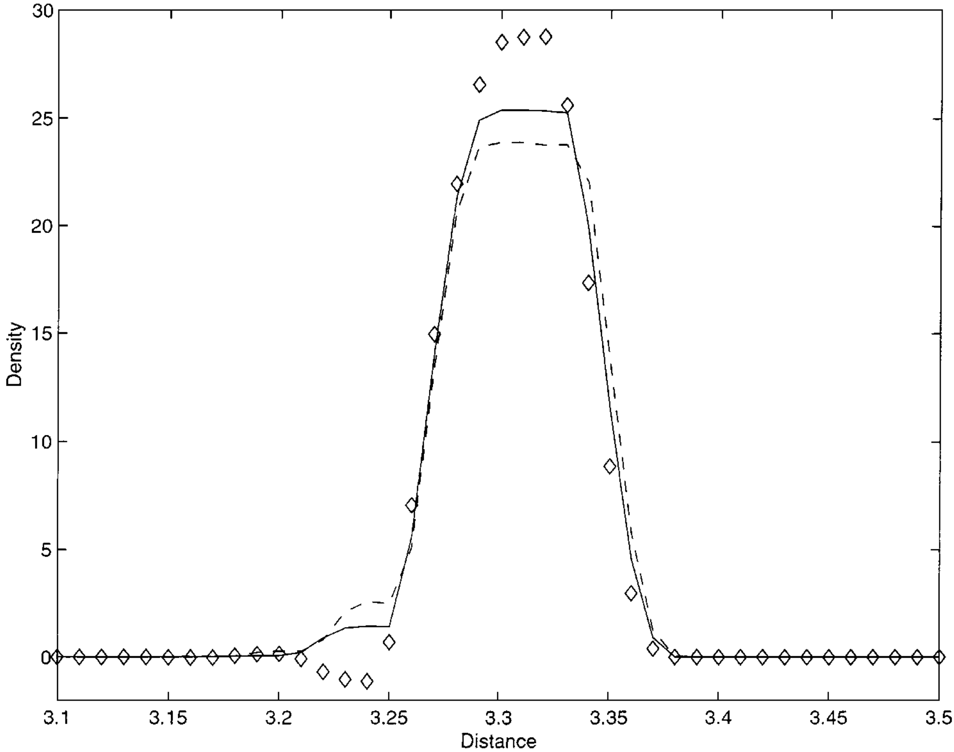

FIG. 1. Performance of the FE-FCT method with various diffusion coefficients c_d: ◇, $c_d = 0.4$ (oscillatory results); - -, $c_d = 1$ (diffusive results); ———self-adjusted c_d (ideal diffusion).

and Morrow [14], Boris and Book [1], Ward [16]). Nevertheless this method becomes very complex in FE and especially in two dimensions and it would be better if it is avoided. The action taken here was to add the optimal diffusion inherent in the upwind scheme to our high-order scheme as a form of mass diffusion. Consider the $(M_c - M_l)$ term in one dimension on a regular grid of length Δx. For the jth node this takes the form

$$(u_{j+1} - 2u_j + u_{j-1})$$

which is a diffusion term. Now the optimal diffusion coefficient, i.e. the diffusion coefficient associated with upwind, is given by Ward [16],

$$D_{uw} = \frac{\Delta t w}{2\Delta x}\left(1 - \frac{\Delta t w}{\Delta x}\right) = \frac{c(1-c)}{2}, \tag{7}$$

where Δt is $t^{n+1} - t^n$, w is the speed, Δx is the mesh spacing at the element-containing node j, and c is the local Courant number at the element-containing node j given by $\Delta t w/\Delta x$. As can be seen from Eq. (7), this optimal diffusion coefficient varies with the mesh size, and time step at each element.

If the diffusion coefficient used in Lohner's method is assumed to be constant in each element, but allowed to vary from element to element as in Eq. (7), then we can introduce the optimal diffusion inherent in the upwind scheme and our low-order scheme will in effect be equivalent to upwind differencing, but with the added advantage that there is no need to resort to the complex upwind method. Furthermore, the diffusion coefficient will automatically be adjustable and, hence, there is no need to optimise c_d, which is one of the disadvantages of Lohner's method.

It is now shown that the addition of the upwind diffusion coefficient to the high-order scheme reduces that to upwind for the simplest case in one dimension with regular mesh of size Δx and constant positive speed w. If we denote $x_j = j\Delta x$, where $j = 1 \cdots K$ and let the flux be $f = wu$ then, at node j, we get the following expression if the high order scheme with the lumped matrix is employed [5],

$$\Delta u_j^n = \Delta t \left[w \frac{\left(u_{j-1}^n - u_{j+1}^n\right)}{2\Delta x} + \frac{w^2 \Delta t}{2} \frac{\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right)}{\Delta x^2} \right] \quad \text{for } j = 2, \ldots, K-1, \tag{8}$$

where

$$\Delta u_j^n = u_j^{n+1} - u_j^n.$$

The addition of the upwind diffusion will be of the form for a regular grid and speed,

$$\frac{w\Delta t}{2\Delta x}\left(1 - \frac{w\Delta t}{\Delta x}\right)\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right). \tag{9}$$

Adding this form of diffusion (Eq. (9)) to the high-order scheme (Eq. (8)) gives

$$\Delta u_j^n = \frac{w\Delta t}{\Delta x}\left(u_{j-1}^n - u_j^n\right) \tag{10}$$

which is, of course, the upwind scheme, and it is always positive. Thus, our method now has the optimum diffusion and the FCT method is expected to give improved performance over the unoptimized global diffusion coefficient used before. In the variable speed, variable mesh case the diffusion coefficient to be used becomes for each node

$$\frac{c_{j+1/2}(1 - c_{j+1/2})}{2}, \tag{11}$$

where

$$c_{j+1/2} = \frac{w_{\text{half}} \Delta t}{\Delta x_{\text{half}}}$$

and w_{half} is the speed and Δx_{half} is the mesh size averaged over the elements containing node j.

4. UPWIND (UW) TESTS

The new low-order method is first of all compared with the upwind differencing scheme in one dimension under different conditions to ensure that the high-order scheme reduces to simple upwind, with the addition of the diffusion coefficient under conditions encountered in gas discharge calculations. These involve constant speed, linearly varying speed, rapidly varying speed, and sign-changing speed.

Figure 2 shows the propagation of a square wave of initial amplitude 10 and width 0.4 cm, starting at $x = 0.4$ cm with a constant speed of $w = 1$ cm/s after a time of 2 s. The results with the two methods are almost identical. The same initial pulse is used in the second test but this time the speed is variable, of the form $w = 5 - x$ cm/s [15]; the pulse is propagated

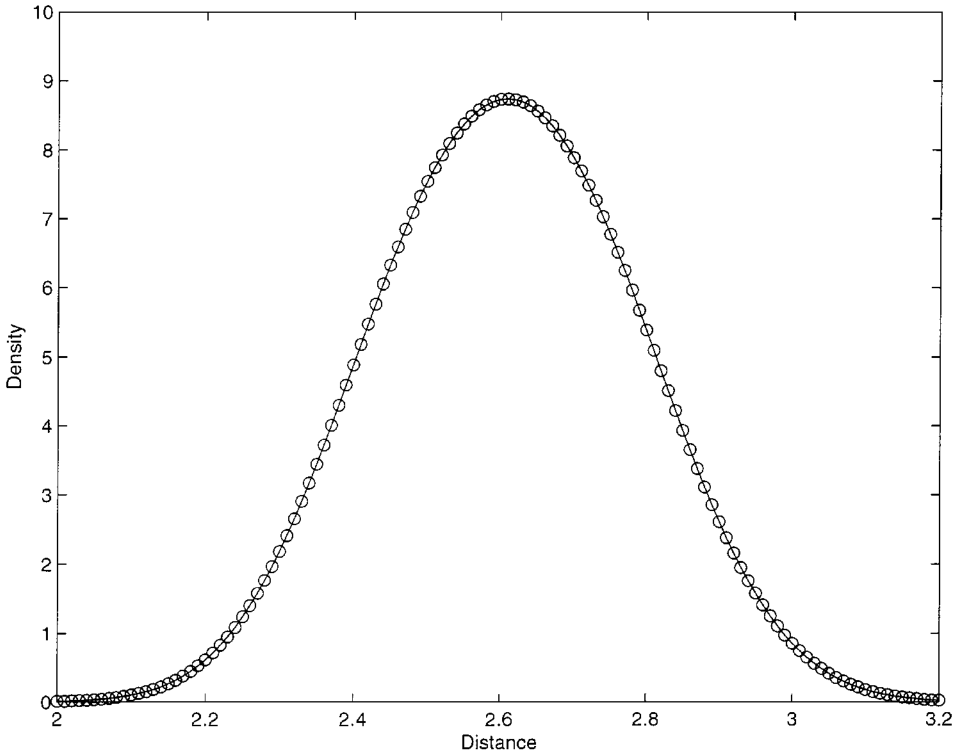

FIG. 2. Square wave test with constant speed using the upwind methods: ○, high-order scheme with upwind (UW) diffusion; solid line, finite difference upwind scheme.

for 1 s with a maximum Courant number of 0.5 and we can see again that the results are almost identical as depicted in Fig. 3. The next test involves a rapidly varying speed of the form $w = 1 + 9\sin^8 \pi x$ cm/s [17]. Such rapidly varying velocities are found in gas discharge calculations. The initial pulse is a square wave of amplitude 10 and width 0.2 cm starting at $x = 0.05$ cm. Figure 4 shows the propagation of the pulse after 0.2363 s. The two methods give again almost undistinguishable results.

Finally the last test is a test where the speed changes sign. The speed is of the form $w = 2.5(2 - x)$ cm/s and the initial pulse starts at 1.8 cm. Figure 5 shows two instances of the pulse, one at 0.5 s and one at 1 s. The results are again in very good agreement which gives confidence in the method and effectively gives upwind differencing results. The next step is to apply that with FCT.

5. FCT-TESTS

The performance of the improved FE-FCT algorithm is compared with Lohner's FE-FCT, the Lax–Wendroff FD-FCT, and the fourth-order FD-FCT. Figure 6 shows for the propagation of a Gaussian pulse of amplitude 10 with a constant speed of $w = 1$ cm/s initially at $x = 0.2$ cm propagated for 0.6 s, that the improved FE-FCT performs quite well with tests other than square waves.

The next test is the propagation of a square wave initially at $x = 0.4$ cm with amplitude 10 and width 0.4 cm with a linearly varying speed of $w = 5 - x$ cm/s. Figure 7 shows a comparison of the improved FE-FCT with the Lax–Wendroff FD-FCT and the fourth-order

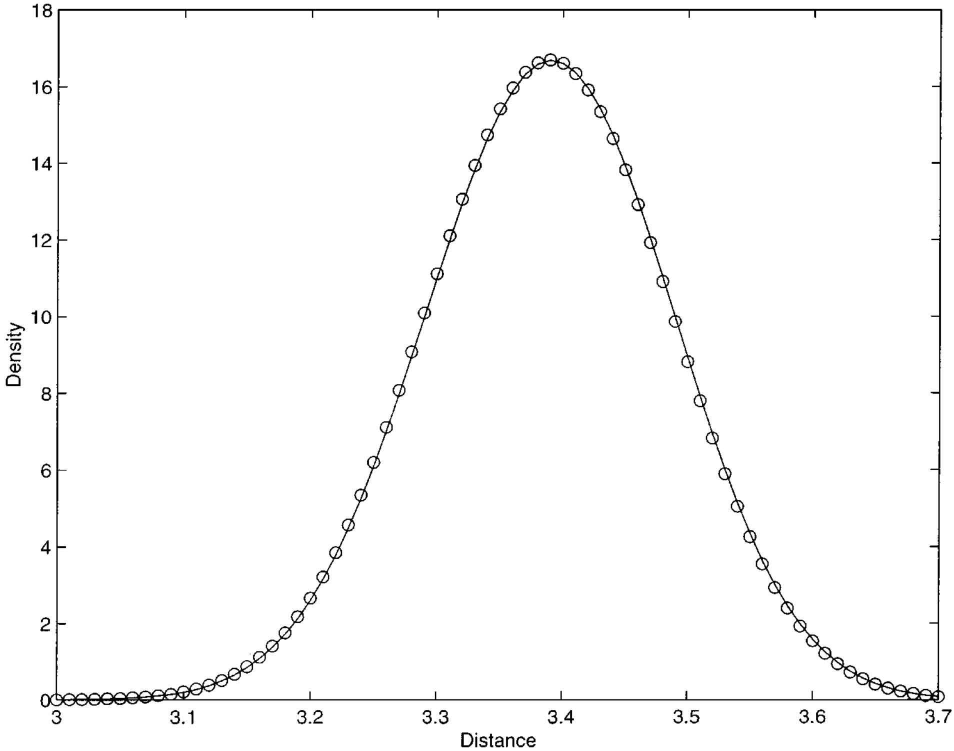

FIG. 3. Square wave test with linearly decreasing speed using the diffusive upwind methods; ○, high-order scheme with UW diffusion; solid line, finite difference upwind scheme.

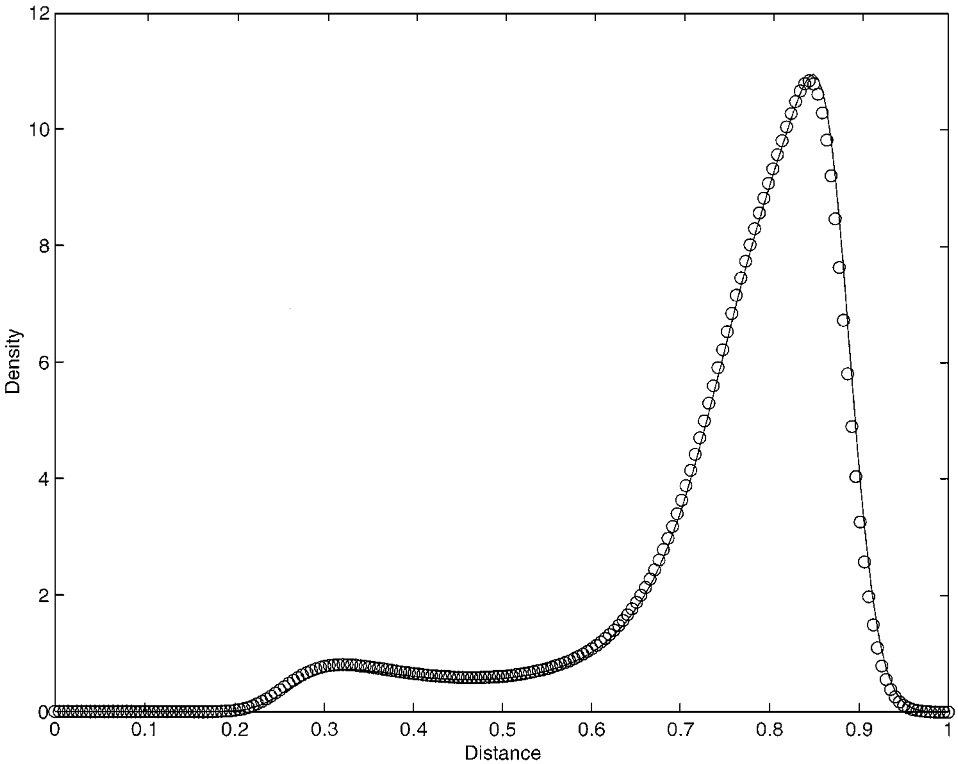

FIG. 4. Square wave test with speed $w = 1 + 9\ \sin^8 \pi x$ cm/s, using the upwind methods: ○, high-order scheme with upwind diffusion; solid line, finite difference upwind scheme.

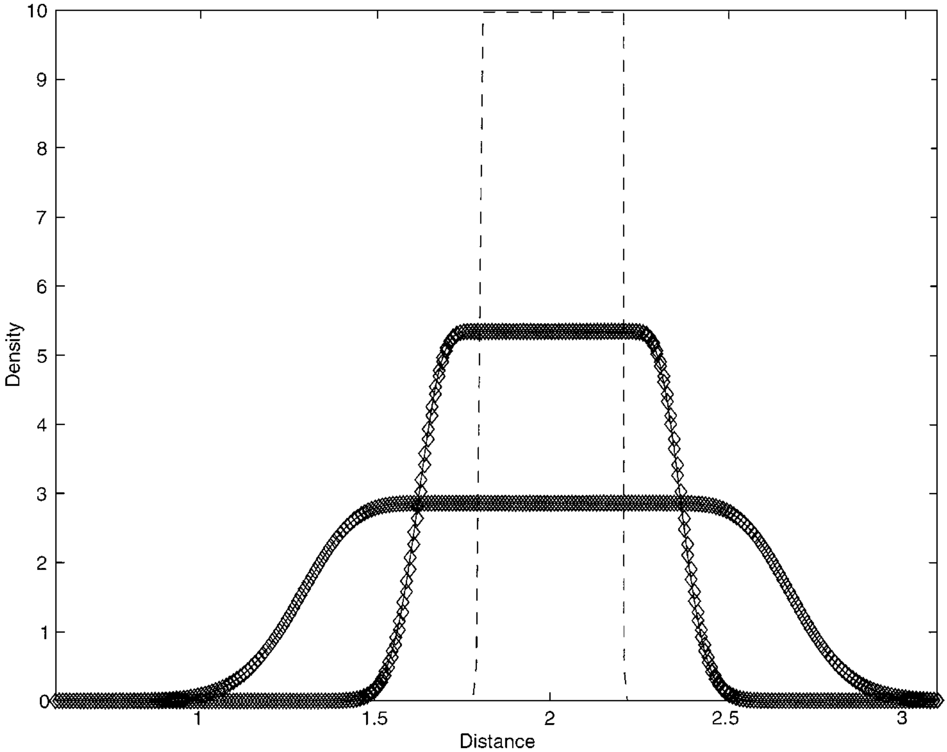

FIG. 5. Square wave test with sign changing speed of the form $w = 2.5(2 - x)$ cm/s using the upwind methods: - -, the intital pulse; ◇, results from the high-order scheme with upwind diffusion at $t = 0.5$ s and $t = 1$ s, solid line, finite difference upwind scheme.

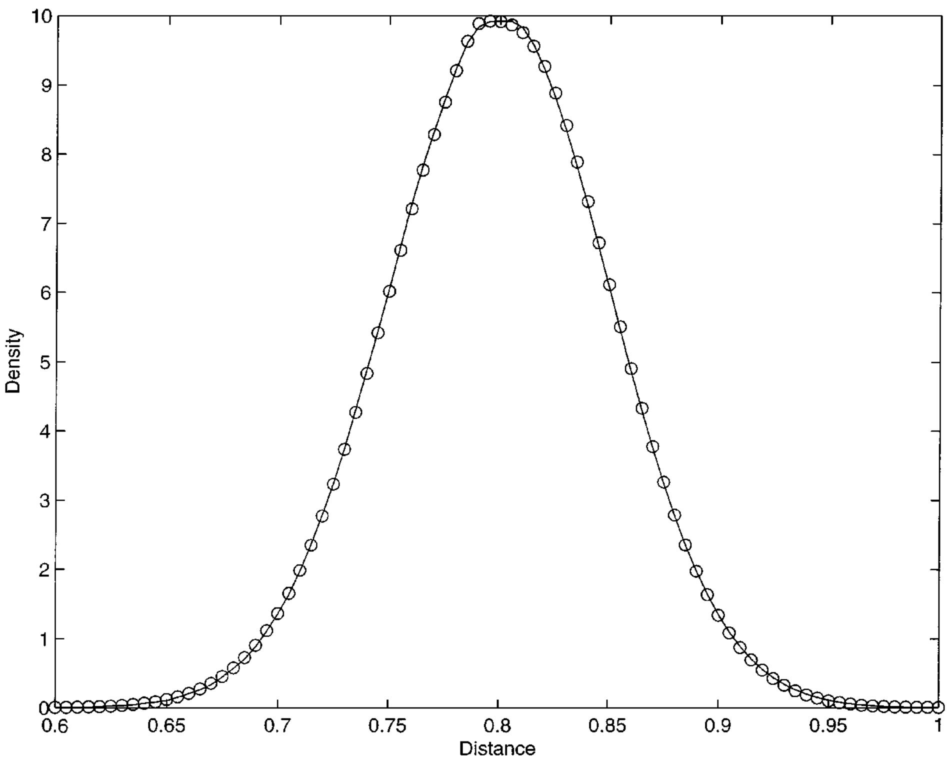

FIG. 6. Comparison of the improved FE-FCT with the fourth-order FD-FCT. The initial wave is a Gaussian pulse in a constant speed field: ○, improved FE-FCT; solid line, fourth-order FD-FCT.

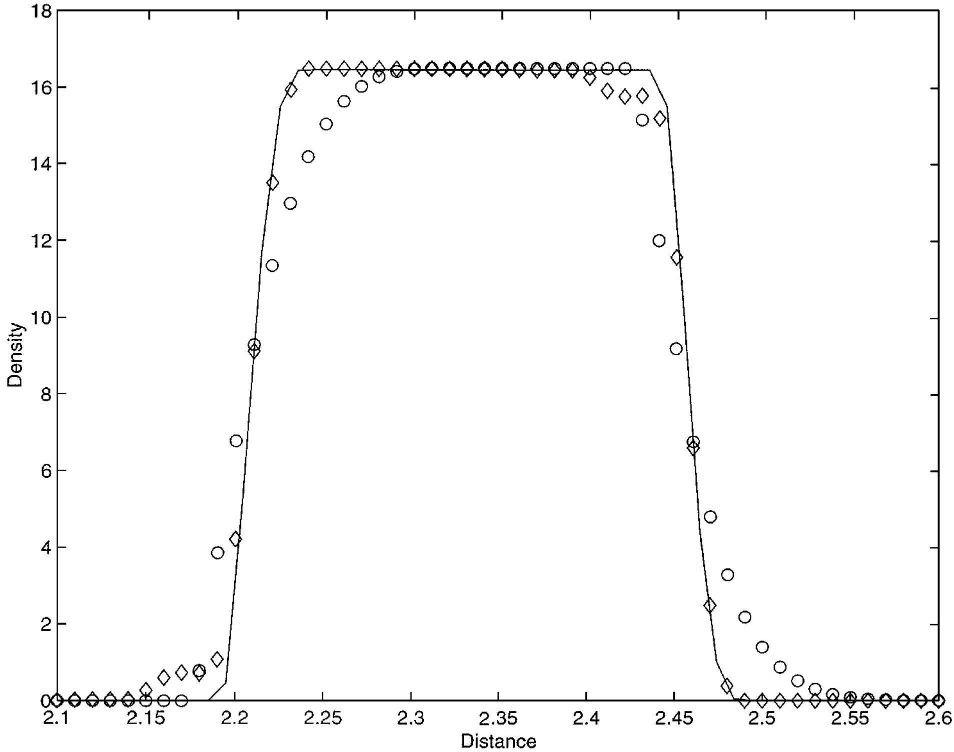

FIG. 7. Comparison of the improved FE-FCT with the Lax–Wendroff FD-FCT and the fourth-order FD-FCT for a square wave with a linearly decreasing speed of the form $w = 5 - x$ cm/s at $t = 0.5$ s: ○, Lax–Wendroff FD-FCT; ◇, improved FE-FCT; solid line, fourth-order FD-FCT.

FD-FCT (Morrow [14]) at 0.5 s, whereas Fig. 8 shows the same pulse at 1 s. It is evident that the FE-FCT (which belongs to the Lax–Wendroff family) is more accurate than the corresponding Lax–Wendroff FD-FCT and this is because of the inclusion of the consistent mass matrix. Furthermore, it gives comparable results to the implicit fourth-order FD-FCT method. Finally, Yousfi's test [17] is carried out and the results obtained with the improved FE-FCT and the fourth-order FD-FCT are shown in Fig. 9. The results are in very good agreement, except at the point where the peak is. This is because the FD scheme is of higher order and so it can resolve the peaks more accurately.

6. FCT-ADVECTION DIFFUSION TESTS

Gas discharge calculations often involve the transport and diffusion of electrons and diffusion can dominate part of the calculation [10, 15], so the code was tested in this kind of conditions. The equation considered this time is

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} = 0, \tag{12}$$

where u is the speed, f is the convective flux, g is the diffusive flux, t is the time, and x is the distance. Given the above equation the solution is advanced from t^n to $t^{n+1} = t^n + \Delta t$ in two steps as before, but with the only difference that the diffusive flux is used only at the corrector step and is evaluated at time t^n. So the two steps become:

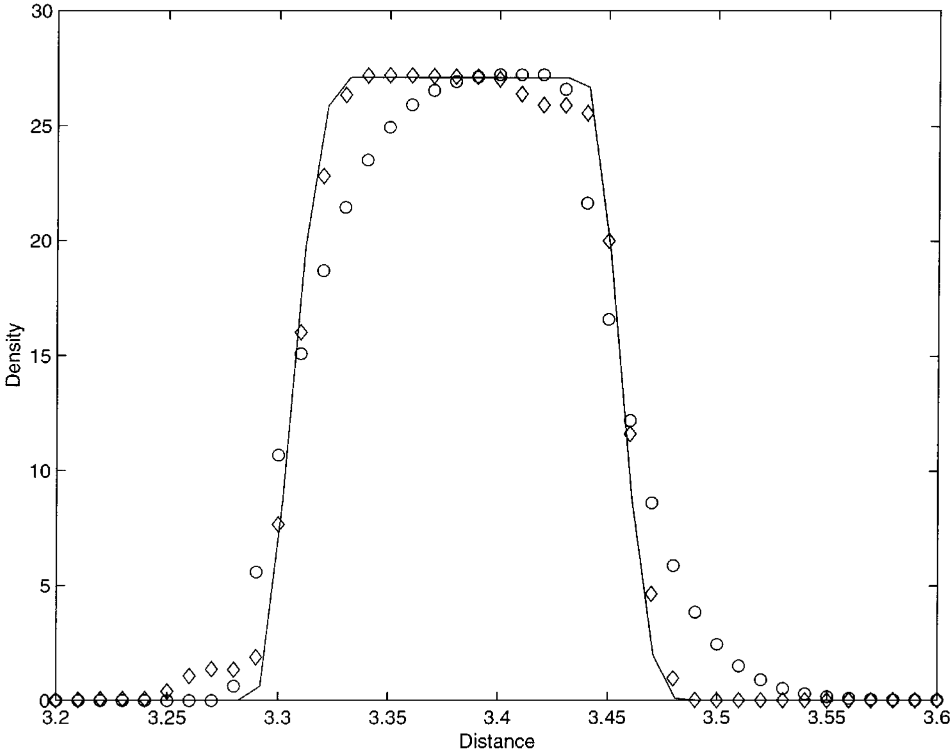

FIG. 8. Comparison of the improved FE-FCT with the Lax–Wendroff FD-FCT and the fourth-order FD-FCT for a square wave with a linearly decreasing speed of the form $w = 5 - x$ cm/s at $t = 1$ s: ○, Lax–Wendroff FD-FCT; ◇, improved FE-FCT; solid line, fourth-order FD-FCT.

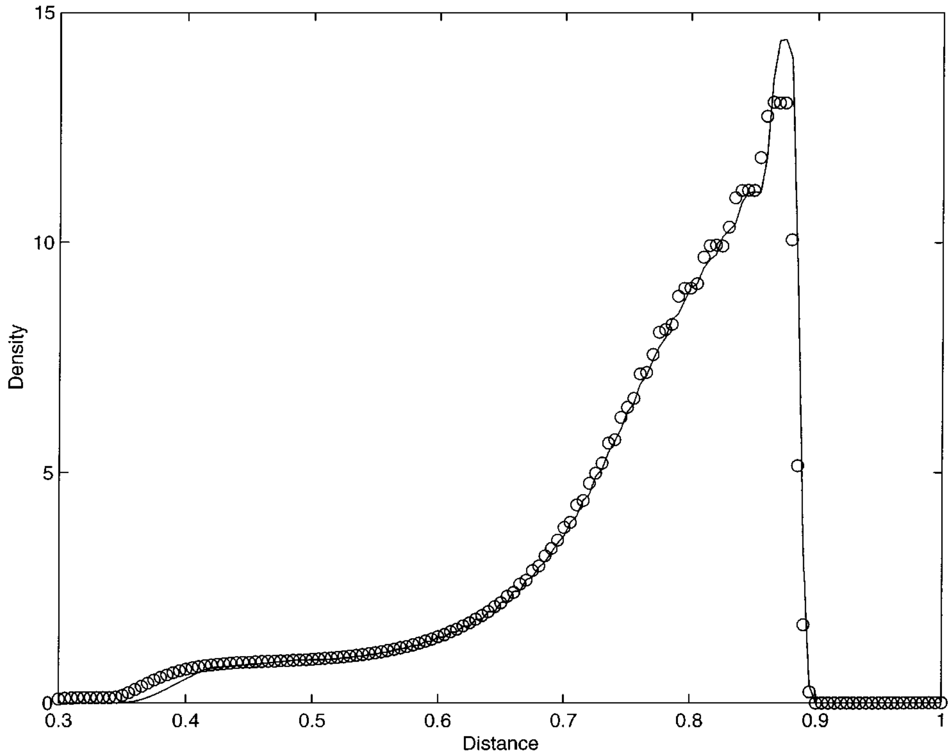

FIG. 9. Comparison of the improved FE-FCT with the fourth-order FD-FCT for a square wave test under a rapidly varying speed field of the form $w = 1 + 9\sin^8 \pi x$ cm/s: ○, improved FE-FCT; solid line, fourth-order FD-FCT.

First step (*advective predictor*),

$$u^{n+1/2} = u^n - \frac{\Delta t}{2}\frac{\partial f^n}{\partial x}. \tag{13}$$

Second step (*corrector step*),

$$\Delta u^n = u^{n+1} - u^n = -\Delta t \frac{\partial f^{n+1/2}}{\partial x} - \Delta t \frac{\partial g^n}{\partial x}. \tag{14}$$

The spatial discretization of Eqs. (13) and (14) is again performed via the classic Galerkin weighted residual method using linear elements.

The method was again tested under various conditions. The first test involved the advection and diffusion of the initial rectangular pulse

$$u(x,0) = \begin{cases} 10, & \text{if } 2b \le x \le 4b,\ b > 0, \\ 0, & \text{if } x < 2b \text{ or } x > 4b. \end{cases}$$

This has the analytic solution [10]

$$u(x,t) = 5\left(\operatorname{erf}\left\{\frac{b - x + wt}{2\sqrt{Dt}}\right\} + \operatorname{erf}\left\{\frac{b + x - wt}{2\sqrt{Dt}}\right\}\right). \tag{15}$$

The values chosen for this test were $w = 2 \times 10^7$ cm/s, $D = 5 \times 10^5$ cm^2/s, and $b = 0.151$ cm. Figure 10 shows the results from the calculation using the improved FE-FCT after 20 ns and these are shown to agree well with the analytical solution.

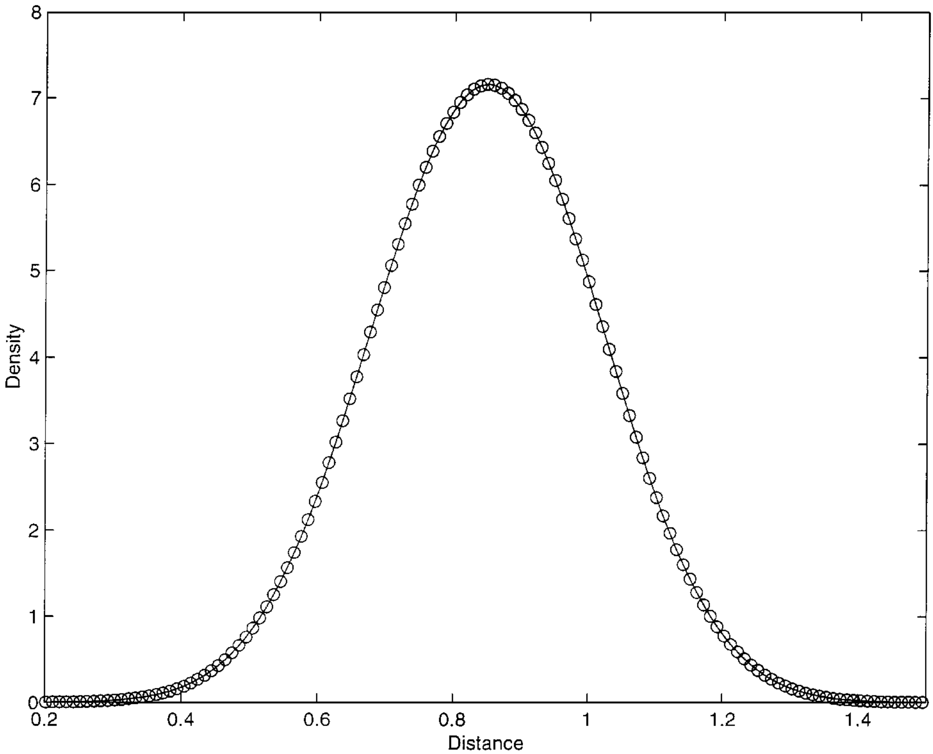

FIG. 10. Square wave test with constant speed $w = 2 \times 10^7$ cm/s and diffusion $D = 5 \times 10^5$ cm^2/s: ○, improved FE-FCT; solid line, analytical solution.

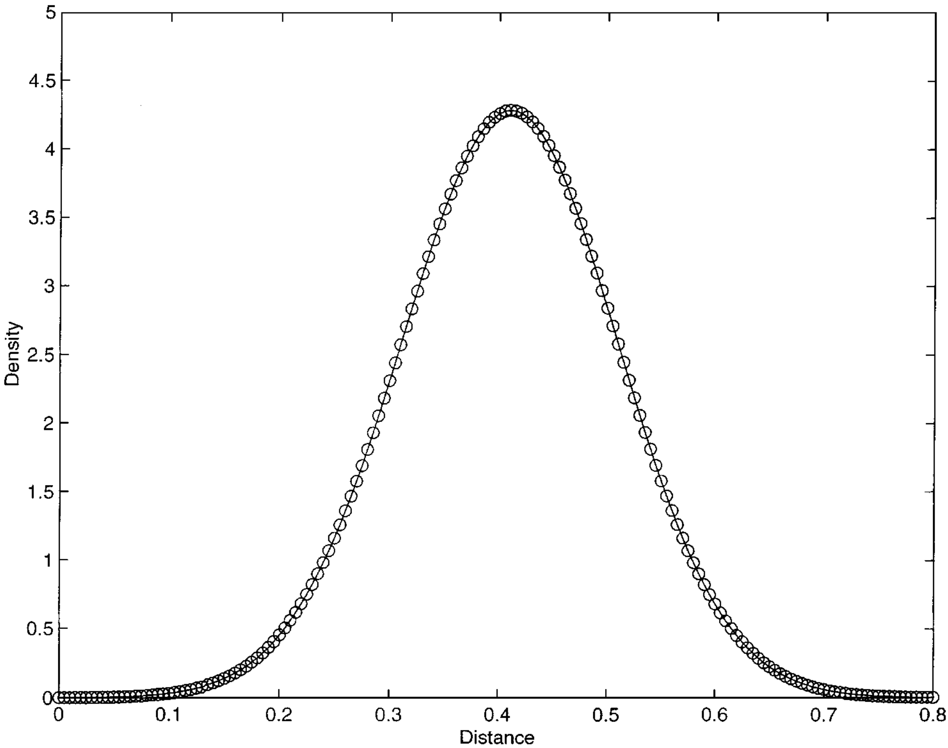

FIG. 11. Gaussian wave test with constant speed $w = 2 \times 10^7$ cm/s and diffusion $D = 5 \times 10^5$ cm^2/s: ○, improved FE-FCT; solid line, analytical solution.

The second test involved propagating a Gaussian pulse of maximum amplitude 10 initially at $x = 0.25$ cm, namely

$$u(x, 0) = 10 \exp\left\{-\frac{(x - 0.25)^2}{4Dt_0}\right\}, \tag{16}$$

with the same speed and diffusion as before and $t_0 = 1.81 \times 10^{-9}$ s. After a time t the distribution becomes [10]

$$u(x, t) = 10(1 + t/t_0)^{-1/2} \exp\left\{\frac{-(x - 0.25 - wt)^2}{4D(t + t_0)}\right\}. \tag{17}$$

Figure 11 compares the results obtained by the improved FE-FCT and the analytical solution after 8 ns and, again, the agreement is very good. Therefore, we are confident that the method works well for advection diffusion problems, as long as we are within the stability criteria which are $c^2 + 2c/p \leq 1$ if the lumped matrix is used or $c^2 + 2c/p \leq \frac{1}{3}$ if the consistent mass matrix is used, where c is the local Courant number and $p = w\Delta x/D$, where D is the local diffusion coefficient [13].

7. APPLICATION TO STREAMER CALCULATIONS

In order to model electrical corona the electron, positive ion, and negative ion continuity equations, including ionization, attachment, recombination, and photoionization are solved simultaneously with Poisson's equation to give distributions of electrons and ion densities and of the electric field [11]. The coupled continuity equations for electrons, positive ions, and negative ions are

$$\frac{\partial N_e}{\partial t} = S + N_e\alpha|W_e| - N_e\eta|W_e| - N_eN_p\beta - \frac{\partial(N_eW_e)}{\partial z} + \frac{\partial}{\partial z}\left(D\frac{\partial N_e}{\partial z}\right) \quad (18)$$

$$\frac{\partial N_p}{\partial t} = S + N_e\alpha|W_e| - N_eN_p\beta - N_nN_p\beta - \frac{\partial(N_pW_p)}{\partial z} \quad (19)$$

$$\frac{\partial N_n}{\partial t} = N_e\eta|W_e| - N_nN_p\beta - \frac{\partial(N_nW_n)}{\partial z}, \quad (20)$$

where t is the time, x is the distance from the anode, N_e, N_p, and N_n are the electron, positive ion, and negative ion densities, respectively, whereas W_e, W_p, and W_n are the electron, positive ion, and negative ion drift velocities, respectively. The symbols, α, η, β, and D denote the ionization, attachment, recombination, and electron diffusion coefficients, respectively. The term S is the source term due to photoionization [11].

Poisson's equation is

$$\nabla(\epsilon_r\nabla\phi) + \frac{e}{\epsilon_0}(N_p - N_n - N_e) = 0, \quad (21)$$

where ϵ_0 is the dielectric constant of free space, ϵ_r is the relative permittivity, e is the electron charge, and ϕ is the electric potential. The details for calculating the transport coefficients

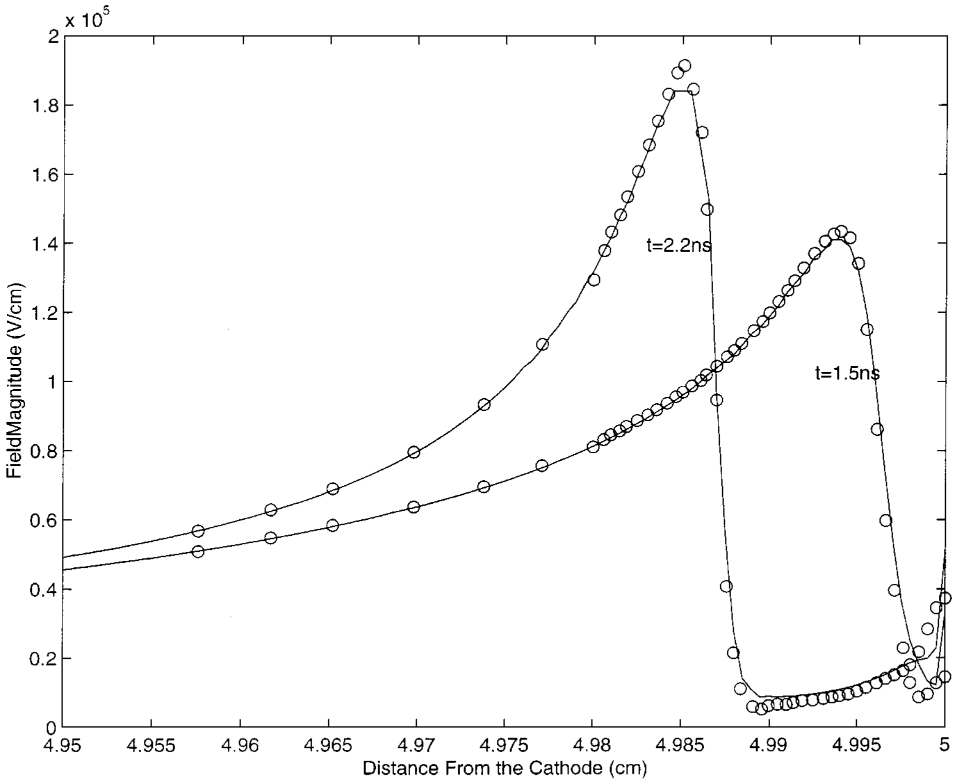

FIG. 12. Field magnitude at 1.5 ns and 2.2 ns in the streamer propagation calculation: ∘, improved FE-FCT; solid line, fourth-order FD-FCT [11] (the anode is at $x = 5$ cm, and the streamer propagation is from right to left).

can be found in other papers [11]. For this case a point plane electrode configuration is used [11] with a gap of 5 cm and tip radius of 0.5 mm, and a voltage of 20 kV is applied at the point electrode. The results for the first few nanoseconds of streamer propagation are shown in Fig. 12, which compares the calculated field using the improved FE-FCT method and the results computed using the computer code developed by Morrow and Lowke [11] who use the fourth-order FD-FCT method to solve the continuity equations. The three continuity equations are solved with the new FE-FCT code developed in one dimension, together with two-dimensional FE for the Poisson's equation as in Morrow and Lowke [11]. The results are in very good agreement. We note the computational savings that can be achieved by using FE over FD. In the FE case 2000 unknowns in an unstructured mesh were used with a very fine resolution near the anode and very coarse resolution at the body of the plasma, whereas in the FD case for the same problem, 20,000 unknowns were present. This shows the advantage of the FE method.

8. CONCLUSION

In this paper an improved FE-FCT algorithm was introduced which is an extension of the very economical and powerful method used by Lohner in fluid mechanics. The diffusion coefficient is optimized by using the effective diffusion coefficient from the upwind differencing algorithm. The diffusion is added as a mass diffusion to the high-order scheme to give a low-order scheme which behaves like upwind differencing, and which has a self-adjustable diffusion coefficient. Thus, the simplicity of Lohner's method is maintained, together with the optimum diffusion of the upwind scheme, whereas to compute an upwind scheme directly would be very complex in two dimensions. The low-order scheme of the improved FE-FCT scheme was shown to give almost identical results, compared with the upwind scheme and the improved FE-FCT was shown to give improved results because the low-order scheme was optimised. The improved FE-FCT scheme is used in gas discharge problems, yielding excellent results. The current work involves extending this improved FE-FCT to two dimensions which will allow us to solve gas discharge problems with arbitrarily shaped electrodes using FE methods. This will result in considerable computational savings over the finite difference method.

ACKNOWLEDGMENT

R. Morrow gratefully acknowledges the support of a Visiting Fellowship, from St. John's College, University of Cambridge, which allowed the development of this new algorithm.

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